Series] possessed the property that was exhibited as belonging to the odd number 197. This was omitted in the copy from some doubt whether it was universally true. Since the paper was read that doubt has been removed, and it turns out that the property belongs not only to *all* the terms of the series 1, 3, 7, 13, &c., but to all odd numbers whatsoever. I am desirous to add to the paper this statement by way of supplement. The property referred to may be thus enunciated:—

Every odd number may be divided into square numbers (not exceeding 4) whose roots (positive or negative) will by their sum or difference [in some form of the roots] give every odd number from 1 to the greatest sum of the roots, which (of course) must always be an odd number.

Or the theorem may be stated in a purely algebraical form, thus:—If there be two equations

$$a^2+b^2+c^2+d^2=2n+1$$
$$a+b+c+d=2r+1,$$

a, b, c, d being each integral or zero, n and r being positive, and r a maximum; then if any positive integer r' (not greater than r) be assumed, it will always be possible to satisfy the pair of equations

$$w^2+x^2+y^2+z^2=2n+1$$
$$w+x+y+z=2r'+1$$

by integral values (positive, negative or zero) of w, x, y, z.

I hope shortly to communicate a proof of the above theorem, independent of any of the usual modes of proving that every odd number is composed of (not exceeding) four square numbers.

Note.—The differences of the roots of 197 were not fully stated in the paper, I add them here:—

197
has 7 forms of roots:—

Forms of roots.	14, 1, 0, 0	
	12, 7, 2, 0	3 ... 7......................17 21
	12, 6, 4, 1	1, 3 ...... 9, 11, 13 19, 21, 23,
	11, 6, 6, 2	1, 3 ...... 9, 13, 21, 25,
	10, 9, 4, 0	3, 5,.................. 15,............... 23
	10, 6, 6, 5	3, 5, 7 .... 15, 17, ... 27.
	9, 8, 6, 4	1, 3, 7 11, 15, 19............... 27.

January 19, 1854.

CHARLES WHEATSTONE, Esq., V.P., in the Chair.

A paper was read, entitled "On the Geometrical Representation of the Expansive Action of Heat, and the Theory of Thermo-dynamic Engines." By William John Macquorn Rankine, Civil Engineer, F.R.S.S. Lond. and Edinb. &c. Received December 5, 1853.

The author remarks, that if abscissæ be measured from an origin

The Royal Society is collaborating with JSTOR to digitize, preserve, and extend access to *Abstracts of the Papers Communicated to the Royal Society of London.* www.jstor.org

of rectangular coordinates, representing the volumes assumed by an elastic substance, and if ordinates, at right angles to those abscissæ, be taken to denote the corresponding expansive pressures exerted by the substance, then any succession of changes of pressure and volume may be represented geometrically by the coordinates of a curve. If such a curve have two extremities, the area included between the curve and the ordinates let fall from its extremities will represent (when positive) the expansive power given out by the substance during the process represented by the curve. Should the curve be closed, returning into itself, so as to denote a cycle of periodical changes of pressure and volume, then will the area, enclosed within the curve, represent (when positive) the expansive power given out during one cycle of changes. This area is positive when increase of volume takes place on the whole at greater pressures than diminution of volume. The area of such a closed curve represents also (when positive) the mechanical equivalent of the heat which permanently disappears, or is converted into expansive power, during a cycle of changes, for were it not so, the sum of energy in the universe would be changed, which is impossible.

As the principles of the expansive action of heat are capable of being presented to the mind more clearly by the aid of diagrams of energy than by means of words and symbols alone, such diagrams are applied, in the present paper, partly to the illustration and demonstration of propositions previously proved by other means, but chiefly to the solution of new questions, especially those relating to the theory of thermo-dynamic engines.

Throughout the whole of this paper, quantities of heat are expressed, not by units of temperature in an unit of weight of water, but by equivalent quantities of mechanical power, stated in foot-pounds according to the ratio established by Mr. Joule's experiments on friction (Phil. Trans. 1850), that is to say, 772 foot-pounds per degree of Fahrenheit, or 1389·6 foot-pounds per Centigrade degree, applied to one pound of liquid water at atmospheric temperatures.

A curve described on a diagram of energy, such that its ordinates represent the pressures of a homogeneous substance corresponding to various volumes of an unit of weight, while the total sensible or actual heat (Q) present in an unit of weight of the substance, is maintained at a constant value (Q_1), may be called the *Isothermal Curve* of Q_1 for the given substance. Its equation is

$$Q=Q_1.$$

If an unit of weight of a substance be allowed to expand, under a pressure equal to its own elasticity, without receiving or emitting heat, its actual heat will diminish during the expansion, and its pressure will diminish more rapidly than it would do if the actual heat were maintained constant. A curve whose coördinates represent this mode of variation of pressure and volume may be called a *Curve of no Transmission of Heat.* For every such curve a certain function of pressure, volume and actual heat, called a *Thermo-dynamic Function* (F), has a constant value (F_A) proper to the particular curve under consideration; whose equation is therefore

$$F=F_A.$$

A curve whose coordinates represent the relation between pressure and volume when the substance is absolutely destitute of heat, may be called the *Curve of Absolute Cold.* It is at once an isothermal curve and a curve of no transmission, and is an asymptote to all the other curves of both those kinds, which approach it indefinitely as the substance expands without limit.

The whole theory of the expansive action of heat is comprehended in the geometrical properties and mutual relations of those two classes of curves; and all those properties and relations are the consequences of and are virtually expressed by the two following theorems:—

THEOREM I.—*The mechanical equivalent of the heat absorbed or given out by a substance in passing from one given state as to pressure and volume to another given state, through a series of states represented by the coordinates of a given curve on a diagram of energy, is represented by the area included between the given curve and two curves of no transmission drawn from its extremities, and indefinitely prolonged in the direction representing increase of volume.*

THEOREM II.—*If across any pair of curves of no transmission on a diagram of energy there be drawn any series of isothermal curves at intervals corresponding to equal differences of actual heat, the series of quadrilateral areas thus cut off from the space between the curves of no transmission will be all equal to each other.*

These two propositions are the geometrical representation of the application, to the particular case of heat and expansive power, of two axioms respecting Energy in the abstract, viz.—

AXIOM I.—*The sum of energy in the Universe is unalterable.*

AXIOM II.—*The effect, in causing transformation of energy, of the whole quantity of actual energy present in a substance, is the sum of the effects of all its parts.*

The application of these axioms to heat and expansive power involves the following

DEFINITION.—*Expansive Heat is a species of actual Energy, the presence of which in a substance affects, and in general increases, its tendency to expand;*—
and this definition, arrived at by induction from experience, is the foundation of the theory of the expansive action of heat.

The first section of the paper is occupied chiefly with the demonstration of the first of the theorems quoted and its immediate consequences, which are applicable to all substances, homogeneous and heterogeneous.

The second section relates to the theory of the expansive action of heat in homogeneous substances.

From the second theorem above quoted, it is deduced, that the area of any quadrilateral bounded above and below by any two isothermal curves, and laterally by two curves of no transmission, is the product of the difference between the two quantities of actual heat proper to the isothermal curves, by the difference between the two thermo-dynamic functions proper to the curves of no transmission, being represented by an expression of this form,

$$(Q_1-Q_2)\,.\,(F_B-F_A).$$

While the area of a figure bounded above by the isothermal curve of Q_1, and laterally by the indefinitely-extended curves of no transmission corresponding to the thermo-dynamic functions F_A, F_B, is represented by

$$Q_1(F_B-F_A).$$

The area of a diagram of energy of any figure is calculated conceiving it to be divided, by a network of isothermal curves and curves of no transmission, into an indefinite number of stripes or quadrilaterals, finding the area of each and adding them by summation or integration. By the aid of these principles various problems are solved.

In the third section the same principles are applied to determine the efficiency of thermo-dynamic engines worked by the expansion and contraction of permanent gases without and with the aid of *economisers* or *regenerators*. The *efficiency* of a thermo-dynamic engine is the proportion of the whole heat communicated to the working substance which is converted into motive power.

The *maximum theoretical efficiency* of a thermo-dynamic engine working between the limits of actual heat Q_1 and Q_2, whether without a regenerator or with a perfect regenerator, is expressed by the fraction

$$\frac{Q_1-Q_2}{Q_1}.$$

A theoretically perfect regenerator does not increase the maximum efficiency between given limits of actual heat, but merely enables that efficiency to be attained with a smaller extent of expansion, and consequently with a smaller engine.

The fourth section treats of the relation between actual heat and temperature, which must be known before the propositions of the preceding sections can be applied to actual substances. Existing experimental data are not yet adequate to the *exact* determination of this relation; but it is considered they are sufficient to show that a relation deduced by the author from the Hypothesis of Molecular Vortices (see Trans. of the Royal Society of Edinburgh, vol. xx.), is at least near enough to the truth for all purposes connected with the computation of the efficiency of thermo-dynamic engines. This relation is expressed by the formula

$$Q=\mathfrak{k}(T+T_0),$$

where T is temperature, measured from the melting-point of ice; T_0, the height of the melting-point of ice above the point of total privation of heat; and $\mathfrak{k}$, the mechanical value of the real specific heat of the substance. According to computations made in 1852 by the author from experiments by Messrs. Thomson and Joule, $T_0 = 272\frac{1}{2}°$ Centigrade $= 490\frac{1}{2}°$ Fahrenheit, a value which may be considered sufficiently correct for practical purposes.

The maximum theoretical efficiency of every conceivable thermodynamic engine receiving heat at the temperature T_1, and giving out heat at the temperature T_2, is

$$\frac{Q_1 - Q_2}{Q_1} = \frac{T_1 - T_2}{T_1 + T_0}.$$

The fourth section concludes with a system of formulæ, illustrated by numerical examples, for computing the power and efficiency of air-engines.

In the fifth section, the principles of the preceding sections are applied to aggregates consisting of heterogeneous substances, or of the same substance in different conditions, especially the aggregate of a liquid and its vapour; and the results are applied to the numerical computation of the theoretical efficiency of steam-engines.

Jan. 26, 1854.

The REV. BADEN POWELL, V.P., in the Chair.

1. A paper was read, entitled "On the Vibrations and Tones produced by the contact of bodies having different Temperatures." By J. Tyndall, Esq., F.R.S. Received Jan. 15, 1854.

The author introduces the subject of his paper by a brief description of the labours which have preceded his own, from the discovery of the phænomenon by Schwartz in 1805 to its revival and further examination by Trevelyan, Faraday, Forbes and Seebeck. The peculiar views of Prof. Forbes, who regards the effects as due to a new species of mechanical agency in heat, were the chief inducement to the resumption of the subject by the author. He examines the ground on which the theory of Prof. Forbes is based, and tests by experiment the general laws at which he has arrived. The first of these laws is, that "the vibrations never take place between substances of the same nature." By converting the cold metal on which the hot rocker is placed into a thin plate, fixing this plate in a vice, and causing the rocker to rest upon the edge of the plate, the author obtained vibrations with iron on iron, brass on brass, copper on copper, silver on silver, zinc on zinc, tin on tin; and thus shows the first law to be untenable. The second general law affirms that "both substances must be metallic." As exceptions to this law, the author adduces experiments made on about twenty non-metallic substances, with which perfectly distinct vibrations were obtained. Among those which signalize themselves by the force and permanence of the rockings they produce, are to be found rock-salt, fluor-spar and rock-crystal. With rockers similar to those described in the paper, and attending to the precautions there dwelt upon, vibrations and musical tones can be obtained without difficulty on these substances.